

Poisson Hierarchy of Discrete Strings

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The Poisson geometry of a discrete string in three dimensional Euclidean space is investigated. For this the Frenet frames are converted into a spinorial representation, the discrete spinor Frenet equation is interpreted in terms of a transfer matrix formalism, and Poisson brackets are introduced in terms of the spinor components. The construction is then generalised, in a self-similar manner, into an infinite hierarchy of Poisson algebras. As an example, the classical Virasoro (Witt) algebra that determines reparametrisation diffeomorphism along a continuous string, is identified as a particular sub-algebra, in the hierarchy of the discrete string Poisson algebra.

A continuous curve in the three dimensional Euclidean space \mathbb{R}^3 is a classic subject of differential geometry [1]. The study of curves in \mathbb{R}^3 is similarly pivotal in physics where plenty of breakthroughs come with strings attached. This includes in particular topics that relate to the theory knots [2] such as topological Chern-Simons theories [3], knotted solitons [4] and exotic exchange statistics [5]. In the present Letter, the Poisson geometry of a *discrete* string in ambient \mathbb{R}^3 is studied. In particular, a novel infinite dimensional, self-similar hierarchy of Poisson bracket algebras is exposed. Such a hierarchy is important, for example in the construction of integrable Hamiltonian models of discrete string dynamics [6, 7]. It also facilitates the numerical study of continuous strings *e.g.* on a discrete lattice. Moreover, the concept of a discrete string is a most useful one to aspects of computer graphics, virtual reality and robotics [8, 9]. Discrete strings also model polymers [10], including biophysical ones from DNA to proteins [11–14].

Our starting point is the description of a discrete string in terms of an open and oriented, piecewise linear polygonal chain $\mathbf{r}(s) \in \mathbb{R}^3$ [15]. The arc length parameter takes values on $s \in [0, L]$ where L is the total length of the string. The vertices \mathcal{D}_i that specify the string are located at the points $\mathbf{r}_i = (\mathbf{r}_0, \dots, \mathbf{r}_n)$ with $\mathbf{r}(s_i) = \mathbf{r}_i$; the endpoints of the string are $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{r}(L) = \mathbf{r}_n$. In addition, the distance of the nearest neighbour vertices \mathcal{D}_i and \mathcal{D}_{i+1} is

$$|\mathbf{r}_{i+1} - \mathbf{r}_i| = s_{i+1} - s_i.$$

Therefore, the nearest neighbour vertices are connected by the line segments

$$\mathbf{r}(s) = \frac{s - s_i}{s_{i+1} - s_i} \mathbf{r}_{i+1} - \frac{s - s_{i+1}}{s_{i+1} - s_i} \mathbf{r}_i, \quad (s_i < s < s_{i+1}).$$

The unit length discrete tangent vector $\mathbf{t}_i = (t_1^i, t_2^i, t_3^i)$ that points from vertex \mathcal{D}_i to vertex \mathcal{D}_{i+1} is defined as

$$\mathbf{t}_i = \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{|\mathbf{r}_{i+1} - \mathbf{r}_i|}. \quad (1)$$

Thus, the vertex \mathcal{D}_k is located at a point

$$\mathbf{r}_k = \sum_{i=0}^{k-1} |\mathbf{r}_{i+1} - \mathbf{r}_i| \mathbf{t}_i. \quad (2)$$

We are interested in the Poisson geometry and the ensuing algebraic structures that can be associated to such a three dimensional discrete string. In lieu of the traditional approach which is based on the discrete Frenet frames in terms of the tangent, normal and binormal vectors, we utilise a two component complex spinor description, with the spinors supported along the string [7]. Then the discrete Frenet equation becomes a two component spinor Frenet equation [7]. Such a spinor based representation of the discrete string geometry has already been found to have

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various conceptual and technical advantages, including the relations between the time evolution of the discrete string and known integrable equations. For more details, see Ref. [7].

We proceed as follows: To each link from vertex \mathcal{D}_i to vertex \mathcal{D}_{i+1} we associate a two component complex spinor

$$\psi_i = \begin{pmatrix} z_1^i \\ z_2^i \end{pmatrix}, \quad (3)$$

where the z_a^i (for $a = 1, 2$ and $i \in \mathbb{Z}$) are complex variables with support on the link. The spinors are related to the unit length tangent vectors by

$$\sqrt{g_i} \mathbf{t}_i = \langle \psi_i, \hat{\sigma} \psi_i \rangle. \quad (4)$$

The $\hat{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the standard Pauli matrices and

$$\sqrt{g_i} \equiv |z_1^i|^2 + |z_2^i|^2 \quad (5)$$

is a metric scale factor. Explicitly,

$$\sqrt{g_i} \begin{pmatrix} t_1^i + it_2^i \\ t_3^i \end{pmatrix} = \begin{pmatrix} 2\bar{z}_1^i z_2^i \\ \bar{z}_1^i z_1^i - \bar{z}_2^i z_2^i \end{pmatrix}. \quad (6)$$

Together with (1) this determines the spinor components (z_1^i, z_2^i) in terms of the vertices \mathcal{D}_i , up to an overall phase. In addition, for each i the conjugate spinor $\bar{\psi}_i$ is defined by introducing the charge conjugation operation \mathcal{C} that acts on ψ_i in the following way

$$\mathcal{C} \psi_i = -i\sigma_2 \psi_i^* = \bar{\psi}_i = \begin{pmatrix} -\bar{z}_2^i \\ \bar{z}_1^i \end{pmatrix}. \quad (7)$$

Observe that

$$\mathcal{C}^2 = -\mathbb{I}$$

and that the two spinors are orthogonal since

$$\langle \psi_i, \bar{\psi}_i \rangle = 0.$$

We combine the spinor components into a 2×2 matrix \mathbf{u}_i as follows:

$$\mathbf{u}_i = \begin{pmatrix} z_1^i & -\bar{z}_2^i \\ z_2^i & \bar{z}_1^i \end{pmatrix}, \quad (8)$$

so that

$$\psi_i = \mathbf{u}_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \bar{\psi}_i = \mathbf{u}_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Our main observation is, that in the case of an infinite number of vertices \mathcal{D}_i this leads to an infinite hierarchy of spinors and ensuing discrete strings and Poisson algebras. Iteratively, and in a self-similar manner; for a finite number of vertices we obtain a finite dimensional sub-hierarchy.

To expose the hierarchy together with its self-similar structure, we start by defining a four-component spinor obtained by combining the two spinors into a Majorana spinor

$$\Psi_i = \begin{pmatrix} -\bar{\psi}_i \\ \psi_i \end{pmatrix}. \quad (9)$$

Indeed, under conjugation by \mathcal{C} this four component spinor transforms according to

$$\Psi_i \longrightarrow \mathcal{C} \Psi_i = -i\sigma_2 \Psi_i^* = \begin{pmatrix} \psi_i \\ \bar{\psi}_i \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{\psi}_i \\ \psi_i \end{pmatrix}$$

where the Pauli matrices σ_a (now) act in the two dimensional space of the spinor components of Ψ_i . In terms of these Majorana spinors, the original discrete spinorial Frenet equation takes the following form

$$\Psi_{i+1} = \mathcal{U}_i \Psi_i. \quad (10)$$

Here \mathcal{U}_i is the ensuing transfer matrix, in the chain of spinors Ψ_i . The self-similar structure emerges when we parametrise \mathcal{U}_i *exactly* in accordance with (8). That is, by setting

$$\mathcal{U}_i = \begin{pmatrix} Z_1^i & -\bar{Z}_2^i \\ Z_2^i & \bar{Z}_1^i \end{pmatrix}. \quad (11)$$

Then a relation between the variables (z_1^i, z_2^i) and (Z_1^i, Z_2^i) is obtained from (10) since

$$\sqrt{g_i} \mathcal{U}_i = \Psi_{i+1} \Psi_i^\dagger. \quad (12)$$

This implies that

$$\begin{aligned} \sqrt{g_i} Z_1^i &= \bar{z}_1^{i+1} z_1^i + \bar{z}_2^{i+1} z_2^i \\ \sqrt{g_i} Z_2^i &= z_1^{i+1} z_2^i - z_2^{i+1} z_1^i. \end{aligned} \quad (13)$$

For each pair of indices (k, l) we introduce the variables $W_1^{k,l}$ and $W_2^{k,l}$ as

$$\begin{aligned} W_1^{k,l} &\stackrel{def}{=} \bar{z}_1^k z_1^l + \bar{z}_2^k z_2^l \\ W_2^{k,l} &\stackrel{def}{=} z_1^k z_2^l - z_2^k z_1^l, \end{aligned} \quad (14)$$

so that in particular,

$$\begin{aligned} W_1^{i+1,i} &\equiv \sqrt{g_i} Z_1^i \\ W_2^{i+1,i} &\equiv \sqrt{g_i} Z_2^i. \end{aligned}$$

Note that the variables (Z_1^i, Z_2^i) are the initiator variables of our self-similar hierarchy, *i.e.* they are the first level variables of the hierarchy. The variables z_1^i and z_2^i comprise the second level variables. The next level of hierarchy then emerges when, in analogy with (13) and (14), we proceed by setting

$$\begin{aligned} \sqrt{g_i} z_1^i &= \bar{\mathfrak{z}}_1^{i+1} \mathfrak{z}_1^i + \bar{\mathfrak{z}}_2^{i+1} \mathfrak{z}_2^i \stackrel{def}{=} w_1^{i+1,i} \\ \sqrt{g_i} z_2^i &= \mathfrak{z}_1^{i+1} \mathfrak{z}_2^i - \mathfrak{z}_2^{i+1} \mathfrak{z}_1^i \stackrel{def}{=} w_2^{i+1,i} \end{aligned} \quad (15)$$

where the metric scale is

$$\sqrt{g_i} = |\mathfrak{z}_1^i|^2 + |\mathfrak{z}_2^i|^2.$$

In analogy with (3), we then introduce the two component spinors

$$\chi_i = \begin{pmatrix} \mathfrak{z}_1^i \\ \mathfrak{z}_2^i \end{pmatrix} \quad \& \quad \bar{\chi}_i = \begin{pmatrix} -\bar{\mathfrak{z}}_2^i \\ \bar{\mathfrak{z}}_1^i \end{pmatrix}$$

which are combined into the following four component Majorana spinors (in accordance with (9)):

$$\mathcal{X}_i = \begin{pmatrix} -\bar{\chi}_i \\ \chi_i \end{pmatrix}. \quad (16)$$

In a self-similar repeat of the previous construction, the Majorana spinors (16) are then related to each other by an equation that has the transfer matrix form

$$\mathcal{X}_{i+1} = \mathbf{u}_i \mathcal{X}_i \quad (17)$$

where \mathbf{u}_i is defined in (8). In analogy with (12), this is the second level transfer matrix which implies that

$$\sqrt{g_i} \mathbf{u}_i = \mathcal{X}_{i+1} \mathcal{X}_i^\dagger.$$

The aforementioned construction can be extended to higher levels, in a straightforward self-similar manner. In this way we obtain the following infinite self-similar hierarchy of variables

$$Z_a^i \xrightarrow{\mathcal{U}_i} z_a^i \xrightarrow{\mathbf{u}_i} \mathfrak{z}_a^i \xrightarrow{\mathbf{v}_i} \dots \quad (a = 1, 2), \quad (18)$$

mapped onto each other by the ensuing transfer matrices $\mathcal{U}_i, \mathbf{u}_i, \mathbf{v}_i, \dots$ as in (10), (17) and so forth; note that the transfer matrices always have the same functional form (8), (11) in the respective variables.

Therefore, for each set of variables in the hierarchy, we can introduce the corresponding piecewise linear discrete string. For this we use the ensuing relation (6) between the variables and the tangent vector of the corresponding string (2). For example, in the case of the Z_a^i (similarly to (4)) we have

$$\sqrt{G_i} \begin{pmatrix} T_1^i + iT_2^i \\ T_3^i \end{pmatrix} = \begin{pmatrix} 2\bar{Z}_1^i Z_2^i \\ \bar{Z}_1^i Z_1^i - \bar{Z}_2^i Z_2^i \end{pmatrix}$$

with the metric scale given by

$$\sqrt{G_i} = |Z_1^i|^2 + |Z_2^i|^2$$

while the vertices of the corresponding string are located at the points

$$\mathbf{R}_k = \sum_{i=0}^{k-1} |\mathbf{R}_{i+1} - \mathbf{R}_i| \cdot \mathbf{T}_i. \quad (19)$$

We also note that the entire hierarchy of strings can be framed, in a self-similar manner, using the following procedure at each level of hierarchy: We recall (8) to introduce the matrices

$$\mathbf{u}_i^{-1} \sigma_3 \mathbf{u}_i = \hat{\mathbf{t}}^i \cdot \boldsymbol{\sigma} \equiv \hat{\mathbf{t}}^i \quad (20)$$

$$\mathbf{u}_i^{-1} \sigma_{\pm} \mathbf{u}_i \equiv \frac{1}{2} \mathbf{u}_i^{-1} (\sigma_1 \pm i\sigma_2) \mathbf{u}_i = \hat{\mathbf{e}}_{\pm}^i \cdot \boldsymbol{\sigma} \equiv \hat{\mathbf{e}}_{\pm}^i, \quad (21)$$

in terms of the transfer matrix \mathbf{u}_i . These matrices obey the $\underline{\text{su}}(2)$ Lie algebra

$$[\hat{\mathbf{t}}^i, \hat{\mathbf{e}}_{\pm}^i] = \pm 2\hat{\mathbf{e}}_{\pm}^i, \quad [\hat{\mathbf{e}}_+^i, \hat{\mathbf{e}}_-^i] = \hat{\mathbf{t}}^i. \quad (22)$$

The $(\hat{\mathbf{t}}^i, \hat{\mathbf{e}}_{\pm}^i)$ then define a generic right-handed orthonormal frame at vertex \mathcal{D}_i . A frame rotation that leaves $\hat{\mathbf{t}}^i$ intact, acts by a $h_i \in \text{U}(1) \subset \text{SU}(2)$ multiplication of \mathbf{u}_i from the left. That is, by letting

$$\mathbf{u}_i \xrightarrow{h_i} h_i \mathbf{u}_i, \quad h_i = e^{i\varphi_i \sigma_3}. \quad (23)$$

we have

$$\hat{\mathbf{t}}^i \xrightarrow{h_i} \mathbf{u}_i^{-1} h_i^{-1} \sigma_3 h_i \mathbf{u}_i \equiv \hat{\mathbf{t}}^i \quad (24)$$

$$\hat{\mathbf{e}}_{\pm}^i \xrightarrow{h_i} \mathbf{u}_i^{-1} h_i^{-1} \sigma_{\pm} h_i \mathbf{u}_i = e^{\pm 2i\varphi_i} \hat{\mathbf{e}}_{\pm}^i. \quad (25)$$

Analogous relations can be introduced, for all levels of the hierarchy, in terms of the ensuing transfer matrices.

We now proceed to reveal our infinite self-similar hierarchy of Poisson algebras, defined in terms of the symplectic structures of the variables in the hierarchy. To start the construction, we impose the following Poisson bracket at the second level of the hierarchy:

$$\{z_a^j, \bar{z}_b^k\} = i\omega(k) \delta_{ab}^{jk}. \quad (26)$$

We assume that all the remaining brackets between the (z_a^i, z_b^j) variables and their conjugates vanish; this clearly defines a symplectic structure, for the local coordinates (z_a^i, z_b^j) . Note that, for the canonical Heisenberg algebra $\omega(k) \equiv 1$ and for the components (6) of the tangent vectors, the Poisson brackets are given by

$$\{\sqrt{g_i} t_a^i, \sqrt{g_j} t_b^j\} = \epsilon_{abc} \delta^{ij} \sqrt{g_i} t_c^i. \quad (27)$$

The self-similar structure then gives us an infinite hierarchy of Poisson algebras, as follows: We simply substitute the ensuing variables in the hierarchy, into the Poisson bracket relation such as (26) which is expressed in terms of the preceding variables in the hierarchy. This yields the brackets between all the variables at all levels of hierarchy, order-by-order. Let us consider, as an example, the algebra that we obtain for the variables defined by the equations (14); note that for this, we have to proceed in the opposite direction, from the second level down to the first. It is

straightforward to show that the (only) non-vanishing brackets of two variables (14), when located at the *same site*, are given by

$$\{W_1^{i+1,i}, \bar{W}_1^{i+1,i}\} = i(\omega(i)\sqrt{g_{i+1}} - \omega(i+1)\sqrt{g_i}) \quad (28)$$

$$\{W_2^{i+1,i}, \bar{W}_2^{i+1,i}\} = i(\omega(i)\sqrt{g_{i+1}} + \omega(i+1)\sqrt{g_i}). \quad (29)$$

Note that (28) vanishes when

$$\frac{\omega(i+1)}{\omega(i)} = \frac{\sqrt{g_{i+1}}}{\sqrt{g_i}}.$$

In addition, (29) becomes the Heisenberg bracket when

$$\omega(i)\sqrt{g_{i+1}} + \omega(i+1)\sqrt{g_i} = 1.$$

Furthermore, for a pair of variables (14) which are located at *different sites* (not necessarily nearest neighbour) the only non-vanishing brackets are the following ones

$$\begin{aligned} \{W_2^{k,l}, \bar{W}_2^{m,n}\} &= i\omega(k)\delta^{km}W_1^{n,l} + i\omega(l)\delta^{ln}W_1^{m,k} - i\omega(l)\delta^{lm}W_1^{n,k} - i\omega(k)\delta^{kn}W_1^{m,l} \\ \{W_1^{k,l}, W_1^{m,n}\} &= i\omega(l)\delta^{lm}W_1^{k,n} - i\omega(k)\delta^{kn}W_1^{m,l} \\ \{W_2^{k,l}, W_1^{m,n}\} &= i\omega(k)\delta^{km}W_2^{n,l} + i\omega(l)\delta^{lm}W_2^{k,n}, \end{aligned}$$

together with their complex conjugated brackets. This is the Poisson algebra, at the first level of the hierarchy. Observe that by definition, due to (14), the $W_i^{k,l}$ (for $i = 1, 2$) satisfy the following identities

$$W_1^{k,l} = \bar{W}_1^{l,k}, \quad W_2^{k,l} = -W_2^{l,k}. \quad (30)$$

Let us finally show how to identify the classical Virasoro generators in terms of these variables. For this we assume that the chain is *infinitely* long. We introduce the combinations

$$L_n^1 = i \sum_{k=-\infty}^{\infty} \bar{z}_1^k z_1^{k-n} \quad \& \quad L_n^2 = i \sum_{k=-\infty}^{\infty} \bar{z}_2^k z_2^{k-n}, \quad (31)$$

where $k \in \mathbb{Z}$. Then the Poisson brackets of the L_n^a are given by

$$\{L_n^a, L_m^b\} = i \sum_{k=-\infty}^{\infty} [\omega(k-m) - \omega(k-n)] \bar{z}_a^k z_a^{k-m-n} \delta^{ab}.$$

By setting $\omega(k) = k$ we find that the variables L_n^a satisfy the classical Virasoro (Witt) algebra. That is,

$$\{L_n^a, L_m^b\} = i(n-m)\delta^{ab}L_{n+m}^a.$$

We also note that

$$L_n^1 + L_n^2 = i \sum_{k=-\infty}^{\infty} W_1^{k,k-n}.$$

Thus, in the infinite hierarchy that we have constructed in terms of *discrete* strings, we have found the algebra of reparametrisations of *continuous* strings, as a sub-algebra. Moreover, we have the following sub-algebra structure

$$\begin{aligned} \{L_n^1 + L_n^2, W_1^{k,l}\} &= -\omega(k)W_1^{k+n,l} + \omega(l)W_1^{k,l-n} \\ \{L_n^1 + L_n^2, W_2^{k,l}\} &= \omega(k)W_2^{k-n,l} + \omega(l)W_2^{k,l-n}. \end{aligned}$$

In summary, we have employed the spinorial formulation of a discrete string in combination with the formalism of discrete Frenet equations to derive an infinite algebraic Poisson hierarchy. As an example, we have shown that the

structure of classical Virasoro (Witt) algebra of continuous strings becomes embedded in this hierarchy. The structure we have revealed, forms a basis for studying the Poisson geometry of discrete strings which is the starting point for constructing integrable structures that model their dynamics in \mathbb{R}^3 .

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- [1] M. Spivak, *A Comprehensive Introduction to Differential Geometry* (Volume Two 3rd Ed.) (Publish or Perish, Inc, Houston, 1999)
 - [2] L. Kauffman, *Knots and Physics*, (World Scientific, Singapore, 1991)
 - [3] E. Witten, Commun. Math. Phys. **121** 351 (1989)
 - [4] L. Faddeev, A.J. Niemi, Nature **387** 58 (1997)
 - [5] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, (World Scientific, Singapore, 1999)
 - [6] S. Hu, Y. Jiang, A.J. Niemi, Phys. Rev. D **87** 105011 (2013)
 - [7] T. Ioannidou, Y. Jiang, A.J. Niemi, Phys Rev D **90** 025012 (2014)
 - [8] A.J. Hanson, *Visualizing Quaternions*, Morgan Kaufmann Elsevier (London) 2006
 - [9] J.B. Kuipers, *Quaternions and Rotation Sequences: a Primer with Applications to Orbits, Aerospace, and Virtual Reality*, Princeton University Press (Princeton) 1999
 - [10] L. Schäfer, *Excluded Volume Effects in Polymer Solutions, as Explained by the Renormalization Group* (Springer Verlag, Berlin, 1999)
 - [11] U.H. Danielsson, M. Lundgren, A.J. Niemi, Phys. Rev. E **82** 021910 (2010)
 - [12] M. Chernodub, S. Hu, A.J. Niemi, Phys. Rev. E **82** 011916 (2010)
 - [13] N. Molkenhuth, S. Hu, A.J. Niemi, Phys. Rev. Lett. **106** 078102 (2011)
 - [14] A.J. Niemi, Theor. Math. Phys. **181** 1235 (2014)
 - [15] S. Hu, M. Lundgren, A.J. Niemi, Phys. Rev. E **83** 061908 (2011)